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BILINEAR MULTIPLIERS OF FUNCTION SPACES WITH WAVELET TRANSFORM IN $L^p_\omega(\mathbb{R}^n)$

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ABSTRACT. Let ω_1, ω_2 be weight functions on $\mathbb{R}^n$. For $1 \le p, q < \infty$, fixed $s \in \mathbb{R}_+$, the space $\left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^n)$ consists of $f \in L^p_{\omega_1}(\mathbb{R}^n)$ such that wavelet transform $W_g f(.,s)$ belongs to $L^q_{\omega_2}(\mathbb{R}^n)$ where $0 \ne g \in S(\mathbb{R}^n)$. This space was defined and investigated by Kulak and Gürkanlı [11]. In this paper using this function space, the vector space of bilinear multipliers is defined in this way. Let ω_1, ω_2, ν_1, ν_2 be slowly increasing weight functions and let ω_3, ν_3 be any weight functions on $\mathbb{R}^n$. Assume that $m(\xi,\eta)$ is a bounded, measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. We define

$$B_m(f,g)(x) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \hat{f}(\xi)\,\hat{g}(\eta)\,m(\xi,\eta)\,e^{2\pi i\langle \xi+\eta, x\rangle}\,d\xi d\eta$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$. We say that $m(\xi,\eta)$ is a bilinear multiplier on $\mathbb{R}^n$ of type $(D(p_i,q_i,\omega_i,\nu_i,s_i))$ if B_m is bounded operator from $\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_{s_1}(\mathbb{R}^n) \times \left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_{s_2}(\mathbb{R}^n)$ to $\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_{s_3}(\mathbb{R}^n)$ where $1 \le p_i, q_i < \infty$, $s_i \in \mathbb{R}^+$ $(i = 1,2,3)$. We denote by $BM[D(p_i,q_i,\omega_i,\nu_i,s_i)]$ the vector space of bilinear multipliers of type $(D(p_i,q_i,\omega_i,\nu_i,s_i))$. In this work, some properties of this space are investigated and some examples of these bilinear multipliers are given.

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1. INTRODUCTION

Throughout this paper we will work on $\mathbb{R}^n$ with Lebesgue measure dx. We denote by $C_c^\infty(\mathbb{R}^n)$ and $S(\mathbb{R}^n)$ the space of infinitely differentiable complex-valued functions with compact support on $\mathbb{R}^n$ and the space of infinitely differentiable complex-valued functions on $\mathbb{R}^n$ rapidly decreasing at infinity, respectively. Let f be a complex valued measurable function on $\mathbb{R}^n$. The translation, character and dilation operators T_x, M_x and D_s are defined by $T_x f(y) = f(y-x)$, $M_x f(y) = e^{2\pi i\langle x,y\rangle} f(y)$ and $D_t^p f(y) = t^{-\frac{n}{p}} f\left(\frac{y}{t}\right)$ respectively for $x, y \in \mathbb{R}^n$, $0 < p, t < \infty$. With this notation out of the way one has, for $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$,

$$(T_x f)\,\hat{}\,(\xi) = M_{-x}\hat{f}(\xi)\,,\ (M_x f)\,\hat{}\,(\xi) = T_x\hat{f}(\xi)\,,\ (D_t^p f)\,\hat{}\,(\xi) = D^{p'}_{t^{-1}}\hat{f}(\xi)\,.$$

For $1 \le p \le \infty$, $L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space. A continuous function ω satisfying $1 \le \omega(x)$ and $\omega(x+y) \le \omega(x)\,\omega(y)$ for $x, y \in \mathbb{R}^n$ will be called a weight function on $\mathbb{R}^n$. If $\omega_1(x) \le \omega_2(x)$ for all $x \in \mathbb{R}^n$, we say that $\omega_1 \le \omega_2$. For $1 \le p \le \infty$, we set

$$L^p_\omega(\mathbb{R}^n) = \{f : f\omega \in L^p(\mathbb{R}^n)\}.$$

It is known that $L^p_\omega(\mathbb{R}^n)$ is a Banach space under the norm

$$\|f\|_{p,\omega} = \|f\omega\|_p = \left\{ \int\limits_{\mathbb{R}^n} |f(x)\,\omega(x)|^p\,dx \right\}^{\frac{1}{p}},\ 1 \le p < \infty,$$

or

$$\|f\|_{\infty,\omega} = \|f\omega\|_\infty = \underset{x\in\mathbb{R}^n}{ess\sup} |f(x)\,\omega(x)|,\ p = \infty.$$

We say that a weight function υ_s polynomial type, if $\upsilon_s(x) = (1+|x|)^s$ for $s \ge 0$. Let f be a measurable function on $\mathbb{R}^n$. If there exists $C > 0$ and $N \in \mathbb{N}$ such that

$$|f(x)| \le C\left(1+|x|^2\right)^N$$

for all $x \in \mathbb{R}^n$, then f is said to be slowly increasing function [6]. It is easy to see that polynomial type weight functions are slowly increasing. For $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is denoted by $\hat{f}$. We know that $\hat{f}$ is a continuous function on $\mathbb{R}^n$. We denote by $M(\mathbb{R}^n)$ the space of bounded regular Borel measures, $M(\omega)$ the space of μ in $M(\mathbb{R}^n)$ such that

$$\|\mu\|_\omega = \int\limits_{\mathbb{R}^n} \omega d|\mu| < \infty.$$

If $\mu \in M(\mathbb{R}^n)$, the Fourier-Stieltjes transform of μ is denoted by $\hat{\mu}$ [18]. Given any fixed $0 \ne g \in S(\mathbb{R}^n)$ (called the wavelet function), the Wavelet transform of a function $f \in L^p(\mathbb{R}^n)$ with respect to g is defined by

$$W_g f(x,s) = |s|^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} f(t)\,\overline{g\left(\frac{t-x}{s}\right)}dt$$

for $x \in \mathbb{R}^n$ and $0 \ne s \in \mathbb{R}$, [5, 19].

Let $0 \ne g \in S(\mathbb{R}^n)$ and ω_1, ω_2 be weight functions on $\mathbb{R}^n$. For $1 \le p, q < \infty$ and fixed $s \in \mathbb{R}_+$, we set

$$\left(D^{p,q}_{\omega_1,\omega_2}\right)_s(\mathbb{R}^n) = \left\{f \in L^p_{\omega_1}(\mathbb{R}^n) \mid W_g f(.,s) \in L^q_{\omega_2}(\mathbb{R}^n)\right\}.$$

It is normed space with the norm $\|f\|_{D^{p,q}_{\omega_1,\omega_2}} = \|f\|_{p,\omega_1} + \|W_g f\|_{q,\omega_2}$. This space is defined and investigated in [11].

Theory of the Fourier bilinear multipliers was started by Coifman and Meyer [4, 17] for smooth symbols. They considered bilinear multipliers under some conditions. Lacey and Thiele considered $m(x) = -isgn(x)$ which leads to the Hilbert transform [14, 15, 16]. They proved that m is (p_1, p_2)-multiplier for each triple (p_1, p_2, p_3) such that $1 < p_1, p_2, p_3 \le \infty$ and $p_3 > \frac{2}{3}$. Then this conclusion was extended by Gilbert and Nahmod [7, 8]. Kening-Stein and Grafakos-Kalton considered $m(x) = \frac{1}{|x|^{1-\alpha}}$ $\left(0 < \alpha < \frac{1}{p_1} + \frac{1}{p_2}\right)$ which leads to the bilinear fractional integral transform, [9, 10]. They proved that m is bilinear multiplier under some conditions. Also Blasco investigated bilinear multipliers and transference method of multipliers for Lebesgue spaces and Lorentz spaces [1, 2, 3]. Then Kulak and Gürkanlı extended these results to weighted Lebesgue spaces, variable exponent Lebesgue spaces, weighted Wiener Amalgam spaces and variable exponent Wiener amalgam spaces [12, 13]. This paper deal with the theory of the bilinear multipliers on the $(D^{p,q}_{\omega_1,\omega_2})_s(\mathbb{R}^n)$ which was studied by Kulak and Gürkanlı [11]. In this present paper bilinear multipliers of type $(D(p_1,q_1,\omega_1,\nu_1,s_1;p_2,q_2,\omega_2,\nu_2,s_2;p_3,q_3,\omega_3,\nu_3,s_3))$ are defined by using this specific function space with wavelet transform.

2. THE BILINEAR MULTIPLIERS SPACE $BM[D(p_1,q_1,\omega_1,\nu_1,s_1;p_2,q_2,\omega_2,\nu_2,s_2;p_3,q_3,\omega_3,\nu_3,s_3)]$

Lemma 2.1. *Let ω_1, ω_2 be a slowly increasing weight functions. Then $C_c^\infty(\mathbb{R}^n)$ is dense in the space $(D^{p,q}_{\omega_1,\omega_2})_s(\mathbb{R}^n)$.*

Proof. We know that $C_c^\infty(\mathbb{R}^n) \subset (D^{p,q}_{\omega_1,\omega_2})_s(\mathbb{R}^n)$ for polynomial weight functions [11]. Similarly the inclusion $C_c^\infty(\mathbb{R}^n) \subset (D^{p,q}_{\omega_1,\omega_2})_s(\mathbb{R}^n)$ is showed for slowly increasing weight functions. Take any $f \in (D^{p,q}_{\omega_1,\omega_2})_s(\mathbb{R}^n)$. Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p_{\omega_1}(\mathbb{R}^n)$ [12], for given $\varepsilon > 0$, there exists a sequence $(u_n)_{n\epsilon\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ and $n_1\epsilon\mathbb{N}$ such that for all $n \geq n_1$

$$\|f - u_n\|_{p,\omega_1} < \frac{\varepsilon}{2}. \tag{2.1}$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $L^q_{\omega_2}(\mathbb{R}^n)$ [12], for given $\varepsilon > 0$, there exists a sequence $(h_n)_{n\epsilon\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ and $n_2\epsilon\mathbb{N}$ such that for all $n \geq n_2$

$$\|W_g f - h_n\|_{q,\omega_2} < \frac{\varepsilon}{2}. \tag{2.2}$$

So $(h_n)_{n\epsilon\mathbb{N}}$ convergences to $W_g f$ in $L^q(\mathbb{R}^n)$. Then there exists a subsequence $(h_{n_k})_{n_k\epsilon\mathbb{N}} \subset (h_n)_{n\epsilon\mathbb{N}}$ such that $(h_{n_k})_{n_k\epsilon\mathbb{N}}$ pointwise convergences to $W_g f$ almost everywhere (a.e). Also, since $(u_n)_{n\epsilon\mathbb{N}}$ convergences to f in $L^p_{\omega_1}(\mathbb{R}^n)$, then there exists subsequence $(u_{n_k})_{n_k\epsilon\mathbb{N}}$ convergences to f in $L^p_{\omega_1}(\mathbb{R}^n)$. So $(u_{n_k})_{n_k\epsilon\mathbb{N}}$convergences to f in $L^p(\mathbb{R}^n)$. Consequently using following inequality, for all $x\epsilon\mathbb{R}^n$

$$\begin{aligned} |W_g u_{n_k}(x) - h_{n_k}(x)| &= |W_g u_{n_k}(x) - h_{n_k}(x)| \\ &\leq \acute{p}^{\left(\frac{n}{\acute{p}} - \frac{n}{2}\right)} \|u_{n_k} - f\|_p \|g\|_{\acute{p}} + |W_g f(x) - h_{n_k}(x)| \ [11], \end{aligned}$$

we obtain $W_g u_{n_k} = h_{n_k}$ (a.e). If we set $n_0 = maks\{n_1, n_2\}$. Combining the inequalities (2.1) and (2.2), for all $\varepsilon > 0$ and $n > n_0$, we have

$$\|f - u_{n_k}\|_{(D^{p,q}_{\omega_1,\omega_2})_s} = \|f - u_{n_k}\|_{p,\omega_1} + \|W_g f - W_g u_{n_k}\|_{q,\omega_2}$$

$$= \|f - u_{n_k}\|_{p,\omega_1} + \|W_g f - h_{n_k}\|_{q,\omega_2} < \varepsilon$$

This completes the proof. □

Let $1 \leq p_1, q_1, p_2, q_2, p_3, q_3 < \infty$, $s_1, s_2, s_3 \epsilon \mathbb{R}^+$ and $\omega_1, \omega_2, \omega_3, \nu_1, \nu_2, \nu_3$ be weight functions on $\mathbb{R}^n$. Assume that $\omega_1, \omega_2, \nu_1, \nu_2$ are slowly increasing functions and $m(\xi, \eta)$ is a bounded, measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. Define

$$B_m(f,g)(x) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \hat{f}(\xi)\,\hat{g}(\eta)\, m(\xi,\eta)\, e^{2\pi i\langle \xi+\eta, x\rangle} d\xi d\eta$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$.

m is said to be a bilinear multiplier on $\mathbb{R}^n$ of type
$D(p_1, q_1, \omega_1, \nu_1, s_1; p_2, q_2, \omega_2, \nu_2, s_2; p_3, q_3, \omega_3, \nu_3, s_3)$ (shortly $(D(p_i, q_i, \omega_i, \nu_i, s_i))$)
, if there exists $C > 0$ such that

$$\|B_m(f,g)\|_{(D^{p_3,q_3}_{\omega_3,\nu_3})_{s_3}} \leq C \|f\|_{(D^{p_1,q_1}_{\omega_1,\nu_1})_{s_1}} \|g\|_{(D^{p_2,q_2}_{\omega_2,\nu_2})_{s_2}}$$

for all $f, g \in C_c^\infty(\mathbb{R}^n)$. That means B_m extends to a bounded bilinear operator from $(D^{p_1,q_1}_{\omega_1,\nu_1})_{s_1}(\mathbb{R}^n) \times (D^{p_2,q_2}_{\omega_2,\nu_2})_{s_2}(\mathbb{R}^n)$ to $(D^{p_3,q_3}_{\omega_3,\nu_3})_{s_3}(\mathbb{R}^n)$.

We denote by $BM[D(p_1, q_1, \omega_1, \nu_1, s_1; p_2, q_2, \omega_2, \nu_2, s_2; p_3, q_3, \omega_3, \nu_3, s_3)]$ (shortly $BM[D(p_i, q_i, \omega_i, \nu_i, s_i)]$) the space of all bilinear multipliers of type $(D(p_i, q_i, \omega_i, \nu_i, s_i))$ and $\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))} = \|B_m\|$.

Lemma 2.2. ***Hölder Inequality for The Space*** $(D^{p,q}_{\omega,\nu})_s(\mathbb{R}^n)$
If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ *then there exists* $C > 0$ *such that*

$$\|fh\|_{(D^{p,p}_{\omega,\omega})_{s_3}} \leq C \|f\|_{(D^{p_1,q_1}_{\omega,\omega_1})_{s_1}} \|h\|_{(D^{p_2,q_2}_{\omega,\omega_2})_{s_2}}$$

where $f \in (D^{p_1,q_1}_{\omega,\omega_1})_{s_1}(\mathbb{R}^n)$ *and* $h \in (D^{p_2,q_2}_{\omega,\omega_2})_{s_2}(\mathbb{R}^n)$.

Proof. Let $f \in (D^{p_1,q_1}_{\omega,\omega_1})_{s_1}(\mathbb{R}^n)$ and $h \in (D^{p_2,q_2}_{\omega,\omega_2})_{s_2}(\mathbb{R}^n)$. It's known that the equality

$$W_g(fh) = (fh) * D_{s_3} g^*.$$

From the this equality and Hölder inequality for Lebesgue spaces, we have

$$\|fh\|_{(D^{p,p}_{\omega,\omega})_{s_3}} = \|fh\|_{p,\omega} + \|W_g(fh)\|_{p,\omega}$$

$$\leq \|f\|_{p_1,\omega} \|h\|_{p_2,\omega} + \|(fh) * D_{s_3} g^*\|_{p,\omega}.$$

On the otherhand since $L^p_\omega(\mathbb{R}^n)$ is Banach module over $L^1_\omega(\mathbb{R}^n)$, using last inequality we obtain

$$\|fh\|_{(D^{p,p}_{\omega,\omega})_{s_3}} \leq \|f\|_{p_1,\omega} \|h\|_{p_2,\omega} + \|fh\|_{p,\omega} \|D_{s_3} g^*\|_{1,\omega}$$

$$\leq \|f\|_{p_1,\omega} \|h\|_{p_2,\omega} + \|f\|_{p_1,\omega} \|h\|_{p_2,\omega} \|D_{s_3} g^*\|_{1,\omega}$$

$$= \left\{1 + \|D_{s_3} g^*\|_{1,\omega}\right\} \|f\|_{p_1,\omega} \|h\|_{p_2,\omega}$$

$$\leq \left\{1 + \|D_{s_3} g^*\|_{1,\omega}\right\} \left\{\|f\|_{p_1,\omega} + \|W_g f\|_{p_1,\omega_1}\right\} \left\{\|h\|_{p_2,\omega} + \|W_g h\|_{p_2,\omega_2}\right\}$$

$$= C \|f\|_{(D^{p_1,q_1}_{\omega,\omega_1})_{s_1}} \|h\|_{(D^{p_2,q_2}_{\omega,\omega_2})_{s_2}}$$

where $C = \left\{1 + \|D_{s_3} g^*\|_{1,\omega}\right\}$. □

The following Theorem is an example to bilinear mutiplier on $\mathbb{R}^n$ of type $(D(p_i, q_i, \omega_i, \nu_i, s_i))$.

Theorem 2.3. *Let* $\frac{1}{q_3}=\frac{1}{p_1}+\frac{1}{p_2}$, ω_3 *be slowly increasing weight function. If*

$K\in L^1_\omega(\mathbb{R}^n)$ *such that* $\omega(y)=\omega_1(y)\,\omega_2(-y)$, $\omega_1=maks\{\omega_3,\nu_1\}$ *and* $\omega_2=maks\{\omega_3,\nu_2\}$ *then* $m(\xi,\eta)=\hat{K}(\xi-\eta)\in BM\left(D\left(p_1,q_1,\omega_3,\nu_1,s_1;p_2,q_2,\omega_3,\nu_2,s_2;q_3,q_3,\omega_3,\omega_3,s_3\right)\right)$. *Furthermore there exists* $C>0$ *such that*

$$\|m\|_{(D(p_1,q_1,\omega_3,\nu_1,s_1;p_2,q_2,\omega_3,\nu_2,s_2;q_3,q_3,\omega_3,\omega_3,s_3))}\le C\,\|K\|_{1,\omega}\,.$$

Proof. Take any $f,h\in C_c^\infty(\mathbb{R}^n)$. We write the equality

$$B_m(f,h)(t)=\int\limits_{\mathbb{R}^n} f(t-y)\,h(t+y)\,K(y)\,dy\ \ [12]\,. \tag{2.3}$$

Also we know that $f(t-y)\in(D^{p_1,q_1}_{\omega_1,\nu_1})_{s_1}(\mathbb{R}^n)$, $h(t+y)\in(D^{p_2,q_2}_{\omega_2,\nu_2})_{s_2}(\mathbb{R}^n)$ [11]. So using (2.3) and Hölder inequality, we have

$$\begin{aligned}\|B_m(f,h)\|_{(D^{q_3,q_3}_{\omega_3,\nu_3})_{s_3}} &= \left\|\int\limits_{\mathbb{R}^n} f(t-y)\,h(t+y)\,K(y)\,dy\right\|_{(D^{q_3,q_3}_{\omega_3,\omega_3})_{s_3}}\\ &\le \int\limits_{\mathbb{R}^n}\|f(t-y)\,h(t+y)\,K(y)\,dy\|_{(D^{q_3,q_3}_{\omega_3,\omega_3})_{s_3}}\,|K(y)|\,dy\end{aligned}$$

$$\le \int\limits_{\mathbb{R}^n} C\,\|f(t-y)\|_{(D^{p_1,q_1}_{\omega_3,\nu_1})_{s_1}}\,\|h(t+y)\|_{(D^{p_2,q_2}_{\omega_3,\nu_2})_{s_2}}\,|K(y)|\,dy. \tag{2.4}$$

On the otherhand we write

$$\|f(t-y)\|_{(D^{p_1,q_1}_{\omega_3,\nu_1})_{s_1}}\le\left\{\omega_3(y)\,\|f\|_{p_1,\omega_3}+\nu_1(y)\,\|W_gf\|_{q_1,\nu_1}\right\} \tag{2.5}$$

and

$$\|h(t+y)\|_{(D^{p_2,q_2}_{\omega_3,\nu_2})_{s_2}}\le\left\{\omega_3(-y)\,\|h\|_{p_2,\omega_3}+\nu_2(-y)\,\|W_gf\|_{q_2,\nu_2}\right\}, \tag{2.6}$$

[11]. So using the inequalities (2.5) and (2.6), since $\omega_1=maks\{\omega_3,\nu_1\}$ and $\omega_2=maks\{\omega_3,\nu_2\}$ we have

$$\|f(t-y)\|_{(D^{p_1,q_1}_{\omega_3,\nu_1})_{s_1}}\le\omega_1(y)\,\|f\|_{(D^{p_1,q_1}_{\omega_3,\nu_1})_{s_1}} \tag{2.7}$$

and

$$\|h(t+y)\|_{(D^{p_2,q_2}_{\omega_3,\nu_2})_{s_2}}\le\omega_2(-y)\,\|h\|_{(D^{p_2,q_2}_{\omega_3,\nu_2})_{s_2}}\,. \tag{2.8}$$

Therefore using the inequalities (2.4), (2.7) and (2.8), we obtain

$$\begin{aligned}\|B_m(f,h)\|_{(D^{q_3,q_3}_{\omega_3,\omega_3})_{s_3}} &\le \int\limits_{\mathbb{R}^n} C\,\|f(t-y)\|_{(D^{p_1,q_1}_{\omega_3,\nu_1})_{s_1}}\,\|h(t+y)\|_{(D^{p_2,q_2}_{\omega_3,\nu_2})_{s_2}}\,|K(y)|\,dy\\ &\le C\int\limits_{\mathbb{R}^n}\omega_1(y)\,\omega_2(-y)\,\|f\|_{(D^{p_1,q_1}_{\omega_3,\nu_1})_{s_1}}\,\|h\|_{(D^{p_2,q_2}_{\omega_3,\nu_2})_{s_2}}\,|K(y)|\,dy\end{aligned}$$

$$= C\,\|f\|_{(D^{p_1,q_1}_{\omega_3,\nu_1})_{s_1}}\,\|h\|_{(D^{p_2,q_2}_{\omega_3,\nu_2})_{s_2}}\,\|K\|_{1,\omega} \tag{2.9}$$

Then $m(\xi,\eta)=\hat{K}(\xi-\eta)$ defines a bilinear multiplier. Finally, using the (2.9), we obtain

$$\begin{aligned}&\|m\|_{(D(p_1,q_1,\omega_3,\nu_1,s_1;p_2,q_2,\omega_3,\nu_2,s_2;q_3,q_3,\omega_3,\omega_3,s_3))}\\ =&\sup_{\|f\|_{(D^{p_1,q_1}_{\omega_3,\nu_1})_{s_1}}\le 1,\ \|h\|_{(D^{p_2,q_2}_{\omega_3,\nu_2})_{s_2}}\le 1}\frac{\|B_m(f,h)\|_{(D^{q_3,q_3}_{\omega_3,\nu_3})_{s_3}}}{\|f\|_{(D^{p_1,q_1}_{\omega_3,\nu_1})_{s_1}}\|h\|_{(D^{p_2,q_2}_{\omega_3,\nu_2})_{s_2}}}\\ \le& C\|K\|_{1,\omega}.\end{aligned}$$

□

Let $1\le p_i,q_i<\infty$, $s_i\epsilon\mathbb{R}^+$ and ω_i,ν_i $(i=1,2,3)$ be weight functions on $\mathbb{R}^n$. Assume that $\omega_1,\omega_2,\nu_1,\nu_2$ are slowly increasing functions. We denote by $\tilde{M}[D(p_1,q_1,\omega_1,\nu_1,s_1;p_2,q_2,\omega_2,\nu_2,s_2;p_3,q_3,\omega_3,\nu_3,s_3)]$

(shortly $\tilde{M}[D(p_i,q_i,\omega_i,\nu_i,s_i)]$) the space of measurable functions $M:\mathbb{R}^n\to\mathbb{C}$ such that $m(\xi,\eta)=M(\xi-\eta)\in BM[D(p_i,q_i,\omega_i,\nu_i,s_i)]$, that is to say

$$B_M(f,g)(x)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\hat{f}(\xi)\,\hat{g}(\eta)\,M(\xi-\eta)\,e^{2\pi i\langle\xi+\eta,x\rangle}d\xi d\eta$$

extends to bounded bilinear map from $(D^{p_1,q_1}_{\omega_1,\nu_1})_{s_1}(\mathbb{R}^n)\times(D^{p_2,q_2}_{\omega_2,\nu_2})_{s_2}(\mathbb{R}^n)$ to $(D^{p_3,q_3}_{\omega_3,\nu_3})_{s_3}(\mathbb{R}^n)$. We denote $\|M\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))}=\|B_M\|$.

Theorem 2.4. *Let ω be slowly increasing weight function. Assume that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}$, $\omega_1\le\omega$, $\omega_2\le\omega$ and $\upsilon(x)=C\left(1+|x|^2\right)^{2N_1}$, $C\ge 0$, $N_1\in\mathbb{N}$ be a weight function. If $\mu\in M(\upsilon)$ and $m(\xi,\eta)=\hat{\mu}(\alpha\xi+\beta\eta)$ for α, $\beta\in\mathbb{R}$, then $m\in BM[(D(p_1,q_1,\omega,\omega_1,s_1;p_2,q_2,\omega,\omega_2,s_2;p,p,\omega,\omega,s_3))]$. Moreover there exists $C>0$ such that*

$$\|m\|_{(D(p_1,q_1,\omega,\omega_1,s_1;p_2,q_2,\omega,\omega_2,s_2;p,p,\omega,\omega,s_3))}\le C\|\mu\|_\upsilon.$$

Proof. Let f, $h\in C_c^\infty(\mathbb{R}^n)$ be given. By Theorem 2.3 in [12], we have

$$B_m(f,h)(t)=\int_{\mathbb{R}^n}f(t-\alpha y)\,h(t-\beta y)\,d\mu(y). \tag{2.10}$$

Also by [11] we have the inequalities

$$\|T_{\alpha y}f\|_{(D^{p_1,q_1}_{\omega,\omega_1})_{s_1}}\le\omega(\alpha y)\|f\|_{p_1,\omega}+\omega_1(\alpha y)\|W_gf\|_{q_1,\omega_1} \tag{2.11}$$

and

$$\|T_{\beta y}h\|_{(D^{p_2,q_2}_{\omega,\omega_2})_{s_2}}\le\omega(\beta y)\|h\|_{p_2,\omega}+\omega_2(\beta y)\|W_gh\|_{q_2,\omega_2}. \tag{2.12}$$

Then by (2.10), (2.11), (2.12) and Hölder inequality,

$$\|B_m(f,g)\|_{(D^{p,p}_{\omega,\omega})_{s_3}}\le\int_{\mathbb{R}^n}\|f(t-\alpha y)\,h(t-\beta y)\|_{(D^{p,p}_{\omega,\omega})_{s_3}}\,d|\mu|(y)$$

$$\leq \int_{\mathbb{R}^n} C \left\| f(t-\alpha y) \right\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \left\| h(t-\beta y) \right\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}} d|\mu|(y)$$

(2.13)

$$\leq \int_{\mathbb{R}^n} C \left\{ \omega(\alpha y) \|f\|_{p_1,\omega} + \omega_1(\alpha y) \|W_g f\|_{q_1,\omega_1} \right\} \left\{ \omega(\beta y) \|h\|_{p_2,\omega} + \omega_2(\beta y) \|W_g h\|_{q_2,\omega_2} \right\} d|\mu|(y)$$

From the assumption $\omega_1 \leq \omega$, $\omega_2 \leq \omega$ and (2.13)

$$\|B_m(f,h)\|_{\left(D^{p,p}_{\omega,\omega}\right)_{s_3}} \leq \int_{\mathbb{R}^n} C\omega(\alpha y)\,\omega(\beta y) \left\{ \|f\|_{p_1,\omega} + \|W_g f\|_{q_1,\omega_1} \right\} \left\{ \|h\|_{p_2,\omega} + \|W_g h\|_{q_2,\omega_2} \right\} d|\mu|(y)$$

$$= \int_{\mathbb{R}^n} C\omega(\alpha y)\,\omega(\beta y) \|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}} d|\mu|(y)$$

$$\leq C \|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}} \int_{\mathbb{R}^n} \omega(\alpha y)\,\omega(\beta y)\, d|\mu|(y) \tag{2.14}$$

Now , suppose that $|\alpha| \leq 1$, $|\beta| \leq 1$. Since ω is slowly increasing weight function, there exists $C_1 \geq 0$ and $N_1 \in \mathbb{N}$ such that

$$\omega(y) \leq C_1 \left(1 + |y|^2\right)^{N_1} \tag{2.15}$$

and then by using (2.15)

$$\int_{\mathbb{R}^n} \omega(\alpha y)\,\omega(\beta y)\, d|\mu|(y) \leq \int_{\mathbb{R}^n} C_1^2 \left(1 + |\alpha|^2 |y|^2\right)^{N_1} \left(1 + |\beta|^2 |y|^2\right)^{N_1} d|\mu|(y)$$

$$\leq C_1^2 \int_{\mathbb{R}^n} \left(1 + |y|^2\right)^{2N_1} d|\mu|(y) = C_1^2 \|\mu\|_\upsilon \tag{2.16}$$

where $\upsilon(y) = C\left(1 + |y|^2\right)^{2N_1}$. Hence by (2.14) and (2.16)

$$\|B_m(f,h)\|_{\left(D^{p,p}_{\omega,\omega}\right)_{s_3}} \leq CC_1^2 \|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}} \|\mu\|_\upsilon \tag{2.17}$$

Thus since $m \in BM[(D(p_1, q_1, \omega, \omega_1, s_1; p_2, q_2, \omega, \omega_2, s_2; p, p, \omega, \omega, s_3))]$ and by (2.17), we obtain

$$\begin{aligned} &\|m\|_{(D(p_1,q_1,\omega,\omega_1,s_1;p_2,q_2,\omega,\omega_2,s_2;p,p,\omega,\omega,s_3))} \\ &= \sup_{\|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \leq 1,\ \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}} \leq 1} \frac{\|B_m(f,h)\|_{\left(D^{p,p}_{\omega,\omega}\right)_{s_3}}}{\|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}}} \\ &\leq CC_1^2 \|\mu\|_\upsilon . \end{aligned}$$

Similarly, if $|\alpha| > 1$,$|\beta| > 1$ then

$$\int_{\mathbb{R}^n} \omega(\alpha y)\,\omega(\beta y)\, d|\mu|(y) < \int_{\mathbb{R}^n} C_1^2 \left(|\alpha|^2 + |\alpha|^2 |y|^2\right)^{N_1} \left(|\beta|^2 + |\beta|^2 |y|^2\right)^{N_1} d|\mu|(y)$$

$$(2.18)\qquad = C_1^2 |\beta|^{2N_1} |\alpha|^{2N_1} \int\limits_{\mathbb{R}^n} \upsilon(y)\, d|\mu|(y) = C_1^2 |\beta|^{2N_1} |\alpha|^{2N_1} \|\mu\|_\upsilon$$

where $\upsilon(y) = C\left(1+|y|^2\right)^{2N_1}$. Therefore by (2.14) and (2.18), we have

(2.19)

$$\|B_m(f,h)\|_{\left(D^{p,p}_{\omega,\omega}\right)_{s_3}} < CC_1^2 |\beta|^{2N_1} |\alpha|^{2N_1} \|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_2}} \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}} \|\mu\|_\upsilon .$$

Hence, we obtain $m \in BM[(D(p_1,q_1,\omega,\omega_1,s_1;p_2,q_2,\omega,\omega_2,s_2;p,p,\omega,\omega,s_3))]$ and by (2.19)

$$\begin{aligned} &\|m\|_{(D(p_1,q_1,\omega,\omega_1,s_1;p_2,q_2,\omega,\omega_2,s_2;p,p,\omega,\omega,s_3))} \\ = &\sup_{\|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \le 1,\ \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}} \le 1} \frac{\|B_m(f,h)\|_{\left(D^{p,p}_{\omega,\omega}\right)_{s_3}}}{\|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}}} \\ < &\ CC_1^2 |\beta|^{2N_1} |\alpha|^{2N_1} \|\mu\|_\upsilon . \end{aligned}$$

Assume that $|\alpha| > 1$, $|\beta| \le 1$. So

$$\int\limits_{\mathbb{R}^n} \omega(\alpha y)\,\omega(\beta y)\, d|\mu|(y) < \int\limits_{\mathbb{R}^n} C_1^2 \left(|\alpha|^2 + |\alpha|^2 |y|^2\right)^{N_1} \left(1+|y|^2\right)^{N_1} d|\mu|(y)$$

$$(2.20)\qquad = C_1^2 |\alpha|^{2N_1} \int\limits_{\mathbb{R}^n} \upsilon(y)\, d|\mu|(y) = C_1^2 |\alpha|^{2N_1} \|\mu\|_\upsilon .$$

Therefore by (2.14) and (2.20), we have

$$(2.21)\qquad \|B_m(f,h)\|_{\left(D^{p,p}_{\omega,\omega}\right)_{s_3}} < CC_1^2 |\alpha|^{2N_1} \|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}} \|\mu\|_\upsilon .$$

Hence, we obtain $m \in BM[(D(p_1,q_1,\omega,\omega_1,s_1;p_2,q_2,\omega,\omega_2,s_2;p,p,\omega,\omega,s_3))]$ and by (2.21)

$$\begin{aligned} &\|m\|_{(D(p_1,q_1,\omega,\omega_1,s_1;p_2,q_2,\omega,\omega_2,s_2;p,p,\omega,\omega,s_3))} \\ = &\sup_{\|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \le 1,\ \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}} \le 1} \frac{\|B_m(f,h)\|_{\left(D^{p,p}_{\omega,\omega}\right)_{s_3}}}{\|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}}} \\ < &\ CC_1^2 |\alpha|^{2N_1} \|\mu\|_\upsilon . \end{aligned}$$

Finally assume that $|\alpha| \le 1$, $|\beta| > 1$. Then

$$\int\limits_{\mathbb{R}^n} \omega(\alpha y)\,\omega(\beta y)\, d|\mu|(y) < \int\limits_{\mathbb{R}^n} C_1 C_2 \left(1+|y|^2\right)^{N_1} \left(|\beta|^2 + |\beta|^2 |y|^2\right)^{N_1} d|\mu|(y)$$

$$(2.22)\qquad = C_1^2 |\beta|^{2N_1} \int\limits_{\mathbb{R}^n} \upsilon(y)\, d|\mu|(y) = C_1^2 |\beta|^{2N_2} \|\mu\|_\upsilon .$$

Therefore by (2.14) and (2.22), we have

$$(2.23)\qquad \|B_m(f,h)\|_{\left(D^{p,p}_{\omega,\omega}\right)_{s_3}} < CC_1^2 |\beta|^{2N_2} \|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}} \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}} \|\mu\|_\upsilon .$$

Hence, we obtain $m \in BM[(D(p_1,q_1,\omega,\omega_1,s_1;p_2,q_2,\omega,\omega_2,s_2;p,p,\omega,\omega,s_3))]$ and by (2.23)

$$\begin{aligned} &\|m\|_{(D(p_1,q_1,\omega,\omega_1,s_1;p_2,q_2,\omega,\omega_2,s_2;p,p,\omega,\omega,s_3))} \\ = &\sup_{\|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_s}\le 1,\ \|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_s}\le 1} \frac{\|B_m(f,h)\|_{\left(D^{p,p}_{\omega,\omega}\right)_{s_3}}}{\|f\|_{\left(D^{p_1,q_1}_{\omega,\omega_1}\right)_{s_1}}\|h\|_{\left(D^{p_2,q_2}_{\omega,\omega_2}\right)_{s_2}}} \\ < &\ CC_1^2|\beta|^{2N_2}\|\mu\|_v . \end{aligned}$$

□

Theorem 2.5. *Let $m \in BM[D(p_i,q_i,\omega_i,\nu_i,s_i)]$. Assume that $\nu_1 \le \omega_1$ and $\nu_2 \le \omega_2$. Then $M_{(\xi_0,\eta_0)}m \in BM[D(p_i,q_i,\omega_i,\nu_i,s_i)]$ for each $(\xi_0,\eta_0)\in\mathbb{R}^{2n}$ and*

$$\left\|M_{(\xi_0,\eta_0)}m\right\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))} \le \omega_1(-\xi_0)\,\omega_2(-\eta_0)\,\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))}.$$

Proof. For any $f,\ h \in C_c^\infty(\mathbb{R}^n)$, we write

$$B_{M_{(\xi_0,\eta_0)}m}(f,h)(x) = B_m(T_{-\xi_0}f, T_{-\eta_0}h)(x) \tag{2.24}$$

by Theorem 2.4 in [12]. Also, the inequalities

$$\|T_{-\xi_0}f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_{s_1}} \le \left\{\omega_1(-\xi_0)\|f\|_{p_1,\omega_1} + \nu_1(-\xi_0)\|W_gf\|_{q_1,\nu_1}\right\}$$

and

$$\|T_{-\eta_0}h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_{s_2}} \le \left\{\omega_2(-\eta_0)\|h\|_{p_2,\omega_2} + \nu_2(-\eta_0)\|W_gh\|_{q_2,\nu_2}\right\}$$

are satisfied [11]. Since $m \in BM[D(p_i,q_i,\omega_i,\nu_i,s_i)]$ and by (2.24), we have

$$\begin{aligned} \left\|B_{M_{(\xi_0,\eta_0)}m}(f,h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_{s_3}} &= \|B_m(T_{-\xi_0}f,T_{-\eta_0}h)\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_{s_3}} \\ &\le \|B_m\|\,\|T_{-\xi_0}f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_{s_1}}\|T_{-\eta_0}h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_{s_2}} \end{aligned}$$

$$\le \|B_m\|\left\{\omega_1(-\xi_0)\|f\|_{p_1,\omega_1}+\nu_1(-\xi_0)\|W_gf\|_{q_1,\nu_1}\right\}\left\{\omega_2(-\eta_0)\|h\|_{p_2,\omega_2}+\nu_2(-\eta_0)\|W_gh\|_{q_2,\nu_2}\right\}$$

$$\le \|B_m\|\left\{\omega_1(-\xi_0)\|f\|_{p_1,\omega_1}+\omega_1(-\xi_0)\|W_gf\|_{q_1,\nu_1}\right\}\left\{\omega_2(-\eta_0)\|h\|_{p_2,\omega_2}+\omega_2(-\eta_0)\|W_gh\|_{q_2,\nu_2}\right\}$$

$$= \omega_1(-\xi_0)\,\omega_2(-\eta_0)\,\|B_m\|\,\|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_{s_1}}\|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_{s_2}} \tag{2.25}$$

and hence $M_{(\xi_0,\eta_0)}m \in BM[D(p_i,q_i,\omega_i,\nu_i,s_i)]$. So by (2.25), we obtain

$$\begin{aligned} &\left\|M_{(\xi_0,\eta_0)}m\right\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))} \\ = &\sup_{\|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_{s_1}}\le 1,\|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_{s_2}}\le 1} \frac{\left\|B_{M_{(\xi_0,\eta_0)}m}(f,h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_{s_3}}}{\|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_{s_1}}\|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_{s_2}}} \\ \le &\ \omega_1(-\xi_0)\,\omega_2(-\eta_0)\,\|B_m\| = \omega_1(-\xi_0)\,\omega_2(-\eta_0)\,\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))}. \end{aligned}$$

□

Lemma 2.6. *If ω and ν are polynomial type weight functions such that $\omega(x) = C_1\left(1+|x|^2\right)^{N_1}$, $\nu(x) = C_2\left(1+|x|^2\right)^{N_2}$ and $f \in (D^{p,q}_{\omega,\nu})_s(\mathbb{R}^n)$, then $D^p_y f \in (D^{p,q}_{\omega,\nu})_s(\mathbb{R}^n)$. Moreover*

$$\left\|D^p_y f\right\|_{\left(D^{p,q}_{\omega,\nu}\right)_s} \le \|f\|_{\left(D^{p,q}_{\omega,\nu}\right)_s}, \qquad \text{if } y \le 1,$$

$$\left\|D^p_y f\right\|_{\left(D^{p,q}_{\omega,\nu}\right)_s} < C\|f\|_{\left(D^{p,q}_{\omega,\nu}\right)_s}, \qquad \text{if } y > 1$$

for some $C > 0$.

Proof. Take any $f \in (D^{p,q}_{\omega,\nu})_s(\mathbb{R}^n)$. For all $x \in \mathbb{R}^n$ and fixed $s \in \mathbb{R}^+$, we set $\frac{t}{y} = u$ $(0 < y < \infty)$

$$W_g\left(D^p_y f\right)(x,s) = |s|^{-\frac{n}{2}} \int_{\mathbb{R}^n} D^p_y f(t)\,\overline{g\left(\frac{t-x}{s}\right)}dt = |s|^{-\frac{n}{2}} \int_{\mathbb{R}^n} |y|^{-\frac{n}{p}} f\left(\frac{t}{y}\right)\overline{g\left(\frac{t-x}{s}\right)}dt$$

$$= |y|^{-\frac{n}{p}}|s|^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(u)\,\overline{g\left(\frac{u.y-x}{s}\right)}y^n du = y^n|y|^{-\frac{n}{p}}|s|^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(u)\,\overline{g\left(\frac{u-\frac{x}{y}}{\frac{s}{y}}\right)}du$$

$$= y^{-\frac{n}{2}+n}|y|^{-\frac{n}{p}}\left|\frac{s}{y}\right|^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(u)\,\overline{g\left(\frac{u-\frac{x}{y}}{\frac{s}{y}}\right)}y^n du = y^{\frac{n}{2}}|y|^{-\frac{n}{p}} W_g(f)\left(\frac{x}{y},\frac{s}{y}\right)$$

$$= y^{\frac{n}{2}} D^p_y W_g(f)(x,s). \tag{2.26}$$

Let $y \le 1$. Also we know that the inequality

$$\left\|D^p_y f\right\|_{p,\omega} \le \|f\|_{p,\omega} \ [12]. \tag{2.27}$$

Since $f \in (D^{p,q}_{\omega,\nu})_s(\mathbb{R}^n)$, we have $W_g(f)(.,s) \in L^q_v(\mathbb{R}^n)$. So by using (2.26) and (2.27)

$$\left\|W_g\left(D^p_y f\right)\right\|_{q,\nu} = \left\|y^{\frac{n}{2}} D^p_y W_g(f)\right\|_{q,\nu}$$

$$= y^{\frac{n}{2}}\left\|D^p_y(W_g f)\right\|_{q,\nu} \le \|W_g f\|_{q,\nu} \tag{2.28}$$

is achieved. So using (2.27) and (2.28), we have

$$\left\|D^p_y f\right\|_{\left(D^{p,q}_{\omega,\nu}\right)_s} = \left\|D^p_y f\right\|_{p,\omega} + \left\|W_g\left(D^p_y f\right)\right\|_{q,\nu} \le \|f\|_{p,\omega} + \|W_g f\|_{q,\nu} = \|f\|_{\left(D^{p,q}_{\omega,\nu}\right)_s}.$$

Let $y > 1$. Again we know that the inequality

$$\left\|D^p_y f\right\|_{p,\omega} < y^{N_1}\|f\|_{p,\omega} \ [12], \tag{2.29}$$

where $\omega(x) = C_1\left(1+|x|^2\right)^{N_1}$. By using (2.26) and (2.29), we write

$$\left\|W_g\left(D^p_y f\right)\right\|_{q,\nu} = \left\|y^{\frac{n}{2}} D^p_y W_g(f)\right\|_{q,\nu}$$

$$= y^{\frac{n}{2}}\left\|D^p_y(W_g f)\right\|_{q,\nu} < y^{\frac{n}{2}+N_2}\|W_g f\|_{q,\nu}, \tag{2.30}$$

where $\nu(x) = C_2\left(1+|x|^2\right)^{N_2}$. Hence combining (2.29) and (2.30), we obtain

$$\left\|D^p_y f\right\|_{\left(D^{p,q}_{\omega,\nu}\right)_s} = \left\|D^p_y f\right\|_{p,\omega} + \left\|W_g\left(D^p_y f\right)\right\|_{q,\nu} < y^{N_1}\|f\|_{p,\omega} + y^{\frac{n}{2}+N_2}\|W_g f\|_{q,\nu}$$

$$\leq C \|f\|_{\left(D^{p,q}_{\omega,\nu}\right)_s}$$

where $C = maks\left\{y^{N_1}, y^{\frac{n}{2}+N_2}\right\}$ □

Theorem 2.7. *Let ω_i, υ_i $(i = 1, 2, 3)$ be polynomial weight functions. If $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$, $0 < y < \infty$ and $m \in BM[D(p_i, q_i, \omega_i, \nu_i, s)]$ then $D^q_y m \in BM[D(p_i, q_i, \omega_i, \nu_i, s)]$. Moreover then*

$$\left\|D^q_y m\right\|_{(D(p_i,q_i,\omega_i,\nu_i,s))} \leq C \|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))}.$$

for some $C > 0$.

Proof. Take any $f, h \in C^\infty_c(\mathbb{R}^n)$. It's known that the equality

$$B_{D^q_y m}(f, h)(y) = D^{p_3}_{y^{-1}} B_m\left(D^{p_1}_y f, D^{p_2}_y h\right)(y) \quad [12]. \tag{2.31}$$

Assume that $y \leq 1$. By from Lemma 2.3, assumption $m \in BM[D(p_i, q_i, \omega_i, \nu_i, s)]$ and the equality (2.31), we write

$$\begin{aligned}
\left\|B_{D^q_y m}(f, h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s} &= \left\|D^{p_3}_{y^{-1}} B_m\left(D^{p_1}_y f, D^{p_2}_y h\right)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s} \\
&\leq C \left\|B_m\left(D^{p_1}_y f, D^{p_2}_y h\right)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s} \\
&\leq C \|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))} \left\|D^{p_1}_y f\right\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_s} \left\|D^{p_2}_y h\right\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_s} \\
&\leq C \|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))} \|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_s} \|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_s}.
\end{aligned}$$

So we obtain

$$\left\|D^q_y m\right\|_{(D(p_i,q_i,\omega_i,\nu_i,s))} \leq C \|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))}.$$

Now suppose that $y > 1$. Again from Lemma 2.3, assumption $m \in BM[D(p_i, q_i, \omega_i, \nu_i, s)]$ and the equality (2.31), we have

$$\left\|B_{D^q_y m}(f, h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s} = \left\|D^{p_3}_{y^{-1}} B_m\left(D^{p_1}_y f, D^{p_2}_y h\right)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s}$$

$$< \left\|B_m\left(D^{p_1}_y f, D^{p_2}_y h\right)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s} \leq \|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))} \left\|D^{p_1}_y f\right\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_s} \left\|D^{p_2}_y h\right\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_s}$$

$$< C \|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))} \|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_s} \|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_s}.$$

Hence we find that

$$\left\|D^q_y m\right\|_{(D(p_i,q_i,\omega_i,\nu_i,s))} < C \|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))}.$$

□

Theorem 2.8. *Let ω_i, υ_i $(i = 1, 2, 3)$ be polynomial type weight functions, $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$ and $m(y\xi, y\eta) = m(\xi, \eta)$, $0 < y < \infty$. Then $m \in BM[D(p_i, q_i, \omega_i, \nu_i, s)]$ if and only if $D^q_y m \in BM[D(p_i, q_i, \omega_i, \nu_i, s)]$*

Proof. Take any $f,\ h\in C_c^{\infty}(\mathbb{R}^n)$. Assume that $y\neq 1$. By from [3], the equality

$$B_m(f,h)=y^{-n\left(\frac{1}{p_3}-\frac{1}{p_1}-\frac{1}{p_2}\right)}B_{D_y^q m}(f,h)$$

written. Suppose that $m\in BM[D(p_i,q_i,\omega_i,\nu_i,s)]$. Then we have

$$\left\|B_{D_y^q m}(f,h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s}=y^{n\left(\frac{1}{p_3}-\frac{1}{p_1}-\frac{1}{p_2}\right)}\left\|B_m(f,h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s}$$

$$\leq y^{n\left(\frac{1}{p_3}-\frac{1}{p_1}-\frac{1}{p_2}\right)}\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))}\|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_s}\|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_s}.$$

Therefore we have $D_y^q m\in BM[D(p_i,q_i,\omega_i,\nu_i,s)]$. Now assume that $D_y^q m\in BM[D(p_i,q_i,\omega_i,\nu_i,s)]$. So similarly we have

$$\left\|B_m(f,h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s}=y^{-n\left(\frac{1}{p_3}-\frac{1}{p_1}-\frac{1}{p_2}\right)}\left\|B_{D_y^q m}(f,h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s}$$

$$\leq y^{-n\left(\frac{1}{p_3}-\frac{1}{p_1}-\frac{1}{p_2}\right)}\left\|D_y^q m\right\|_{(D(p_i,q_i,\omega_i,\nu_i,s))}\|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_s}\|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_s}.$$

Hence we obtain $m\in BM[D(p_i,q_i,\omega_i,\nu_i,s)]$. □

Theorem 2.9. *Let $m\in BM[D(p_i,q_i,\omega_i,\nu_i,s_i)]$. Assume that $\nu_1\leq\omega_1$ and $\nu_2\leq\omega_2$. If $\Phi\in L^1_\omega\left(\mathbb{R}^{2n}\right)$ such that $\omega(u,v)=\omega_3(u)\nu_3(v)$, then $\hat{\Phi}m\in BM[D(p_i,q_i,\omega_i,\nu_i,s_i)]$ and*

$$\left\|\hat{\Phi}m\right\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))}\leq\|\Phi\|_{1,\omega}\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))}.$$

Proof. Let $\Phi\in L^1_\omega\left(\mathbb{R}^{2n}\right)$. Take any $f,\ h\in C_c^{\infty}(\mathbb{R}^n)$. It is known by Proposition 2.5 in [3].

$$B_{\hat{\Phi}m}(f,h)(x)=\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\Phi(u,v)\,B_{M_{(-u,-v)}m}(f,h)(x)\,dudv.$$

Since $m\in BM[D(p_i,q_i,\omega_i,\nu_i,s_i)]$ and by Theorem 2.3, we have $M_{(-u,-v)}m\in BM[D(p_i,q_i,\omega_i,\nu_i,s_i)]$ and

$$\begin{aligned}&\left\|M_{(-u,-v)}m\right\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))}\\ \leq\ &\omega_1(u)\,\omega_2(v)\,\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))}.\end{aligned}$$

Then,

$$\left\|B_{\hat{\Phi}m}(f,h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_{s_3}}\leq\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\left\|\Phi(u,v)\,B_{M_{(-u,-v)}m}(f,h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_{s_3}}dudv$$

$$\leq\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|\Phi(u,v)|\left\|M_{(-u,-v)}m\right\|\|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_{s_1}}\|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_{s_2}}dudv$$

$$\leq\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}|\Phi(u,v)|\,\omega_3(u)\,\nu_3(v)\,\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))}\|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_{s_1}}\|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_{s_2}}dudv$$

$$=\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))}\|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_{s_1}}\|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_{s_2}}\|\Phi\|_{1,\omega}. \tag{2.32}$$

Thus from (2.32), we obtain $\hat{\Phi}m \in BM[D(p_i, q_i, \omega_i, \nu_i, s_i)]$ and

$$\left\|\hat{\Phi}m\right\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))} \le \|\Phi\|_{1,\omega}\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s_i))}.$$

□

Theorem 2.10. *Let ω_i, υ_i $(i = 1, 2, 3)$ be polynomial type weight functions and let $m \in BM[D(p_i, q_i, \omega_i, \nu_i, s)]$. If $\Psi \in L^1\left(\mathbb{R}^+, t^{-\frac{2n}{q}}dt\right)$ such that $\frac{2}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3}$ then*

$$m_\Psi(\xi,\eta) = \int_0^\infty m(t\xi, t\eta)\Psi(t)\,dt \in BM[D(p_i, q_i, \omega_i, \nu_i, s)].$$

Moreover,

$$\|m_\Psi\|_{(D(p_i,q_i,\omega_i,\nu_i,s))} < C\|\Psi\|_{L^1\left(\mathbb{R}^+,t^{-\frac{2n}{q}}dt\right)}\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))}$$

for some $C > 0$.

Proof. Take any f, $h \in C_c^\infty(\mathbb{R}^n)$. It's known that

$$B_{m_\Psi}(f,h)(x) = \int_0^\infty B_{D^q_{t^{-1}}m}(f,h)\Psi(t)\,t^{-\frac{2n}{q}}dt\ \ [12].$$

Since $m \in BM[D(p_i, q_i, \omega_i, \nu_i, s)]$ and by Theorem 2.4, we observe that $D^q_{t^{-1}}m \in BM[(p_i, q_i, \omega_i, \nu_i, s)]$ and

$$\|B_{m_\Psi}(f,h)(x)\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s} \le \int_0^\infty \left\|B_{D^q_{t^{-1}}m}(f,h)\right\|_{\left(D^{p_3,q_3}_{\omega_3,\nu_3}\right)_s}|\Psi(t)|\,t^{-\frac{2n}{q}}dt$$

$$= \int_0^\infty \left\|D^q_{t^{-1}}m\right\|_{(D(p_i,q_i,\omega_i,\nu_i,s))}\|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_s}\|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_s}|\Psi(t)|\,t^{-\frac{2n}{q}}dt$$

$$\le C\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))}\|f\|_{\left(D^{p_1,q_1}_{\omega_1,\nu_1}\right)_s}\|h\|_{\left(D^{p_2,q_2}_{\omega_2,\nu_2}\right)_s}\|\Psi\|_{L^1\left(\mathbb{R}^+,t^{-\frac{2n}{q}}dt\right)}.$$

Hence $m_\Psi \in BM[D(p_i, q_i, \omega_i, \nu_i, s)]$ and

$$\|m_\Psi\|_{(D(p_i,q_i,\omega_i,\nu_i,s))} < C\|\Psi\|_{L^1\left(\mathbb{R}^+,t^{-\frac{2n}{q}}dt\right)}\|m\|_{(D(p_i,q_i,\omega_i,\nu_i,s))}.$$

□

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